

# ON THE FIXED POINTS OF NONEXPANSIVE MAPPINGS IN DIRECT SUMS OF BANACH SPACES

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**ABSTRACT.** We show that if a Banach space  $X$  has the weak fixed point property for nonexpansive mappings and  $Y$  has the generalized Gossez-Lami Dozo property or is uniformly convex in every direction, then the direct sum  $X \oplus Y$  with a strictly monotone norm has the weak fixed point property. The result is new even if  $Y$  is finite-dimensional.

## 1. INTRODUCTION

One of the central themes in metric fixed point theory is the existence of fixed points of nonexpansive mappings. Recall that a mapping  $T : C \rightarrow C$  is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all  $x, y \in C$ . A Banach space  $X$  is said to have the fixed point property (FPP) if every nonexpansive self-mapping defined on a nonempty bounded closed and convex set  $C \subset X$  has a fixed point. A Banach space  $X$  is said to have the weak fixed point property (WFPP) if every nonexpansive self-mapping defined on a nonempty weakly compact and convex set  $C \subset X$  has a fixed point.

Fixed point theory for nonexpansive mappings has its origins in the 1965 existence theorems of F. Browder, D. Göhde and W. A. Kirk. The most general of them, Kirk's theorem [21] asserts that all Banach spaces with weak normal structure have WFPP. Recall that a Banach space  $X$  has weak normal structure if  $r(C) < \text{diam } C$  for all weakly compact convex subsets  $C$  of  $X$  consisting of more than one point, where  $r(C) = \inf_{x \in C} \sup_{y \in C} \|x - y\|$  is the Chebyshev radius of  $C$ . In 1981, Alspach [1] showed an example of a nonexpansive mapping defined on a weakly compact convex subset of  $L_1[0, 1]$  without a fixed point, and Maurey [27] used the Banach space ultrapower construction to prove FPP for all reflexive subspaces of  $L_1[0, 1]$  as well as WFPP for  $c_0$  and  $H^1$ . Maurey's method has been applied by numerous authors to obtain several fixed point results. In 2003, García Falset, Lloréns Fuster and Mazcuñan Navarro [12] solved a long-standing problem in the theory by proving FPP for all uniformly nonsquare Banach spaces. Quite recently, Lin [25] showed the first example of a nonreflexive Banach space with FPP, and Domínguez Benavides [8] proved that every reflexive Banach space can be renormed to have FPP, thus solving other classical problems in metric fixed point theory. It is still unknown whether reflexivity (or even

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superreflexivity) implies the fixed point property. For a detailed exposition of metric fixed point theory we refer the reader to [2, 14, 15].

The problem of whether FPP or WFPP is preserved under direct sums of Banach spaces has been studied since the 1968 Belluce–Kirk–Steiner theorem [3], which states that a direct sum of two Banach spaces with normal structure, endowed with the maximum norm, also has normal structure. In 1984, Landes [23] showed that normal structure is preserved under a large class of direct sums including all  $\ell_p^N$ -sums,  $1 < p \leq \infty$ , but not under  $\ell_1^N$ -direct sums (see [24]). Nowadays, there are many results concerning permanence properties of conditions which imply normal structure, see [7, 26, 29] and references therein. Several recent papers consider the general case, but always under additional geometrical assumptions, see [4–6, 9, 10, 18, 19, 28, 30].

Recently, two general fixed point theorems in direct sums were proved in [28]. In the present paper we are able to remove additional assumptions imposed on the space  $X$  in that paper. We show in Section 3 that if a Banach space  $X$  has WFPP and  $Y$  has the generalized Gossez-Lami Dozo property introduced in [16] (see Section 2 for the definition), then the direct sum  $X \oplus Y$  with respect to a strictly monotone norm has WFPP. The result is new even if  $Y$  is a finite-dimensional space and in this case answers a question of Khamsi [20] for strictly monotone norms. Some consequences of the main theorem are presented in Section 4. In particular, we prove that  $X \oplus Y$  has WFPP whenever  $X$  has WFPP and  $Y$  is uniformly convex in every direction.

## 2. PRELIMINARIES

Let us recall several properties of a Banach space  $X$  which are sufficient for weak normal structure. The normal structure coefficient is given by

$$N(X) = \inf \{ \text{diam } A / r(A) \},$$

where the infimum is taken over all bounded convex sets  $A \subset X$  with  $\text{diam } A > 0$  and  $r(A)$  denotes the Chebyshev radius of  $A$  (relative to itself). Assuming that  $X$  does not have the Schur property, we put

$$\text{WCS}(X) = \inf \{ \text{diam}_a(x_n) / r_a(x_n) \},$$

where the infimum is taken over all sequences  $(x_n)$  which converge to 0 weakly but not in norm. Here

$$\text{diam}_a(x_n) = \lim_{n \rightarrow \infty} \sup_{k, l \geq n} \|x_k - x_l\|$$

denotes the asymptotic diameter of  $(x_n)$  and

$$r_a(x_n) = \inf \left\{ \limsup_{n \rightarrow \infty} \|x_n - x\| : x \in \overline{\text{conv}}(x_n)_{n=1}^\infty \right\}$$

denotes the asymptotic radius of  $(x_n)$ . We say that a Banach space  $X$  has uniform normal structure if  $N(X) > 1$  and weak uniform normal structure (or satisfies Bynum's condition) if  $\text{WCS}(X) > 1$ . A weaker property was

introduced in [16]. A Banach space  $X$  is said to have the generalized Gossez-Lami Dozo property (GGLD, in short) if

$$\limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \|x_n - x_m\| > 1$$

whenever  $(x_n)$  converges weakly to 0 and  $\lim_{n \rightarrow \infty} \|x_n\| = 1$ . It is known that  $N(X) > 1 \Rightarrow \text{WCS}(X) > 1 \Rightarrow \text{GGLD} \Rightarrow \text{weak normal structure}$  and that the GGLD property is equivalent to the so-called property asymptotic (P) (see, e.g., [29]).

Recall that a norm  $\|\cdot\|$  on  $\mathbb{R}^2$  is said to be monotone if

$$\|(x_1, y_1)\| \leq \|(x_2, y_2)\| \quad \text{whenever } 0 \leq x_1 \leq x_2, 0 \leq y_1 \leq y_2.$$

A norm  $\|\cdot\|$  is said to be strictly monotone if

$$\begin{aligned} \|(x_1, y_1)\| < \|(x_2, y_2)\| \quad \text{whenever } 0 \leq x_1 \leq x_2, 0 \leq y_1 < y_2 \\ \text{or } 0 \leq x_1 < x_2, 0 \leq y_1 \leq y_2. \end{aligned}$$

It is easy to see that  $\ell_p^2$ -norms,  $1 \leq p < \infty$ , are strictly monotone.

Let  $Z$  be a normed space  $(\mathbb{R}^2, \|\cdot\|_Z)$ . We shall write  $X \oplus_Z Y$  for the  $Z$ -direct sum of Banach spaces  $X, Y$  with the norm

$$\|(x, y)\| = \|(\|x\|, \|y\|)\|_Z,$$

where  $(x, y) \in X \times Y$ . The following lemma was proved in [28, Lemma 4]. Similar arguments can be found in [11, 29].

**Lemma 2.1.** *Let  $X \oplus_Z Y$  be a direct sum of Banach spaces  $X, Y$  with respect to a strictly monotone norm. Assume that  $Y$  has the GGLD property, the vectors  $w_n = (x_n, y_n) \in X \oplus_Z Y$  tend weakly to 0 and*

$$\lim_{n, m \rightarrow \infty, n \neq m} \|w_n - w_m\| = \lim_{n \rightarrow \infty} \|w_n\|.$$

*Then  $\lim_{n \rightarrow \infty} \|y_n\| = 0$ .*

### 3. THE MAIN THEOREM

The following observation is crucial for many fixed point existence theorems for nonexpansive mappings. Assume that there exists a nonexpansive mapping  $T : C \rightarrow C$  without a fixed point, where  $C$  is a nonempty weakly compact convex subset of a Banach space  $X$ . Let

$$\mathcal{F} = \{K \subset C : K \text{ is nonempty, closed, convex and } T(K) \subset K\}.$$

From the weak compactness of  $C$ , any decreasing chain of elements in  $\mathcal{F}$  has a nonempty intersection which belongs to  $\mathcal{F}$ . By the Kuratowski–Zorn lemma, there exists a minimal (in the sense of inclusion) convex and weakly compact set  $K \subset C$  which is invariant under  $T$  and which is not a singleton. Let  $(x_n)$  be an approximate fixed point sequence for  $T$  in  $K$ , i.e.,  $\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0$ . It was proved independently by Goebel [13] and Karlovitz [17] that

$$\lim_{n \rightarrow \infty} \|x_n - x\| = \text{diam } K$$

for every  $x \in K$ . A fruitful approach to the fixed point problem is to use this special feature of minimal invariant sets.

Let  $T : K \rightarrow K$  be a nonexpansive mapping, where  $K$  is a weakly compact convex subset of the direct sum  $X \oplus_Z Y$  with respect to a strictly monotone norm, which is minimal invariant for  $T$ .

Under suitable conditions imposed on the Banach space  $Y$ , we will show that  $K$  is isometric to a subset of  $X$ , thus proving, that  $X \oplus_Z Y$  has WFPP whenever  $X$  does. To this end we first construct, for every integer  $k \geq 1$ , an appropriate family of subsets of  $K$  as follows.

**Lemma 3.1.** *Assume that  $T : K \rightarrow K$  is a nonexpansive mapping defined on a weakly compact convex subset  $K$  of  $X \oplus_Z Y$ , which is minimal invariant for  $T$  and  $\text{diam } K = 1$ . Let  $(w_n) = ((w'_n, w''_n))$  be an approximate fixed point sequence for  $T$  in  $K$  weakly converging to  $(0, 0) \in K$  and  $\lim_{n \rightarrow \infty} \|w''_n\| = 0$ . Fix an integer  $k \geq 1$  and a sequence  $(\varepsilon_n)$  in  $(0, 1)$ . Then there exist a subsequence  $(v_n) = (x_n, y_n)$  of  $(w_n)$  and a family  $\{D_j^i\}_{1 \leq j \leq k, i \geq 1}$  of relatively compact convex subsets of  $K$  such that*

- (i)  $\|Tv_i - v_i\| < \varepsilon_i$ ,
- (ii)  $\|y_i\| < \varepsilon_i$ ,
- (iii)  $\|v_i - z\| > 1 - \varepsilon_i$  for all  $z \in D_k^{i-1}$ ,
- (iv)  $D_1^i = \text{conv}(D_1^{i-1} \cup \{v_i\})$ ,
- (v)  $D_{j+1}^i = \text{conv}(D_j^i \cup T(D_j^i))$ ,

for every  $i \geq 1$  and  $1 \leq j \leq k - 1$  ( $D_1^0 = D_k^0 = \emptyset$ ).

*Proof.* We proceed by induction on  $i$ . Since  $\|Tw_n - w_n\|$  and  $\|w''_n\|$  converge to 0, we can choose  $v_1 = w_{n_1} = (x_1, y_1)$  in such a way that  $\|Tv_1 - v_1\| < \varepsilon_1$  and  $\|y_1\| < \varepsilon_1$ . Let us put

$$D_1^1 = \{v_1\}$$

and, for a given relatively compact convex set  $D_j^1, 1 \leq j < k$ ,

$$D_{j+1}^1 = \text{conv}(D_j^1 \cup T(D_j^1)).$$

By induction on  $j$ , we obtain a family  $\{D_1^1, \dots, D_k^1\}$  of relatively compact convex subsets of  $K$  which satisfies the desired conditions.

Now suppose that we have chosen  $n_1 < \dots < n_l$  ( $l \geq 1$ ),  $v_i = w_{n_i} = (x_i, y_i), 1 \leq i \leq l$ , and a family  $\{D_j^i\}_{1 \leq j \leq k, 1 \leq i \leq l}$  of relatively compact convex subsets of  $K$  such that the conditions (i)-(v) are satisfied for every  $1 \leq i \leq l$  and  $1 \leq j \leq k - 1$ . Then, there exist  $n_{l+1} > n_l$ ,  $v_{l+1} = w_{n_{l+1}} = (x_{l+1}, y_{l+1})$  such that  $\|Tv_{l+1} - v_{l+1}\| < \varepsilon_{l+1}$ ,  $\|y_{l+1}\| < \varepsilon_{l+1}$  and  $\|v_{l+1} - z\| > 1 - \varepsilon_{l+1}$  for all  $z \in D_k^l$  (the last inequality follows from the Goebel-Karlovitc lemma and the relative compactness of  $D_k^l$ ). Let us put

$$D_1^{l+1} = \text{conv}(D_1^l \cup \{v_{l+1}\})$$

and, for a given relatively compact convex set  $D_j^{l+1}, 1 \leq j < k$ ,

$$D_{j+1}^{l+1} = \text{conv}(D_j^{l+1} \cup T(D_j^{l+1})).$$

Then, by induction with respect to  $j$ , we obtain a family  $\{D_1^{l+1}, \dots, D_k^{l+1}\}$  of relatively compact convex subsets of  $K$  which satisfies the desired conditions.

By induction on  $i$ , the lemma follows.  $\square$

We are now going to prove that for a sequence  $(\varepsilon_n(k))$ , if  $u = (a, b) \in \bigcup_{i=1}^{\infty} D_k^i(k)$  and  $k$  is large, then  $b$  is close to 0. We need the following lemma.

**Lemma 3.2.** *Assume that a sequence  $(v_n) = (x_n, y_n)$  and a family  $\{D_j^i\}_{1 \leq j \leq k, i \geq 1}$  of relatively compact convex subsets of  $K$  are given as in Lemma 3.1. Then, for every  $1 \leq j \leq k$ ,  $i \geq 1$  and  $u \in D_j^{i+1}$ , there exists  $z \in D_j^i$  such that*

$$\|z - u\| + \|u - v_{i+1}\| \leq \|z - v_{i+1}\| + 3(j-1)\varepsilon_{i+1}.$$

*Proof.* Fix  $i \geq 1$ . We proceed by induction with respect to  $j$ . For  $j = 1$  and  $u \in D_1^{i+1} = \text{conv}(D_1^i \cup \{v_{i+1}\})$  there exists  $z \in D_1^i$  such that

$$\|z - u\| + \|u - v_{i+1}\| = \|z - v_{i+1}\|.$$

Now fix  $1 \leq j < k$  and suppose that for every  $u \in D_j^{i+1}$  there exists  $z \in D_j^i$  such that

$$\|z - u\| + \|u - v_{i+1}\| \leq \|z - v_{i+1}\| + 3(j-1)\varepsilon_{i+1}. \quad (1)$$

Let

$$u \in D_{j+1}^{i+1} = \text{conv}(D_j^{i+1} \cup T(D_j^{i+1})).$$

Consider three cases.

1° The inductive step is obvious if  $u \in D_j^{i+1}$ .

2° Let  $u \in T(D_j^{i+1})$ . Then  $u = T\bar{u}$  for some  $\bar{u} \in D_j^{i+1}$  and, by assumption, there exists  $\bar{z} \in D_j^i$  such that

$$\|\bar{z} - \bar{u}\| + \|\bar{u} - v_{i+1}\| \leq \|\bar{z} - v_{i+1}\| + 3(j-1)\varepsilon_{i+1}.$$

Let  $z = T\bar{z} \in D_{j+1}^i \subset D_k^i$ . Then

$$\begin{aligned} \|z - u\| + \|u - v_{i+1}\| &\leq \|T\bar{z} - T\bar{u}\| + \|T\bar{u} - Tv_{i+1}\| + \|Tv_{i+1} - v_{i+1}\| \\ &< \|\bar{z} - \bar{u}\| + \|\bar{u} - v_{i+1}\| + \varepsilon_{i+1} \leq \|\bar{z} - v_{i+1}\| + (3j-2)\varepsilon_{i+1} \\ &< \|z - v_{i+1}\| + (3j-1)\varepsilon_{i+1}, \end{aligned} \quad (2)$$

since, by (i),  $\|Tv_{i+1} - v_{i+1}\| < \varepsilon_{i+1}$  and, by (iii),  $\|z - v_{i+1}\| > 1 - \varepsilon_{i+1} \geq \|\bar{z} - v_{i+1}\| - \varepsilon_{i+1}$  (diam  $K = 1$ ).

3° Let  $u = \sum_{s=1}^t \lambda_s u_s$  for some  $u_s \in D_j^{i+1} \cup T(D_j^{i+1})$ ,  $\lambda_s \in [0, 1]$ ,  $1 \leq s \leq t \in \mathbb{N}$ ,  $\sum_{s=1}^t \lambda_s = 1$ . Then, by (1) or (2), there exist  $z_1, \dots, z_t \in D_{j+1}^i$  such that

$$\|z_s - u_s\| + \|u_s - v_{i+1}\| \leq \|z_s - v_{i+1}\| + (3j-1)\varepsilon_{i+1}, 1 \leq s \leq t.$$

Hence

$$\begin{aligned} \left\| \sum_{s=1}^t \lambda_s z_s - u \right\| + \|u - v_{i+1}\| &\leq \sum_{s=1}^t \lambda_s \|z_s - v_{i+1}\| + (3j-1)\varepsilon_{i+1} \\ &\leq 1 + (3j-1)\varepsilon_{i+1} < \left\| \sum_{s=1}^t \lambda_s z_s - v_{i+1} \right\| + 3j\varepsilon_{i+1}, \end{aligned}$$

since, by (iii),  $\left\| \sum_{s=1}^t \lambda_s z_s - v_{i+1} \right\| > 1 - \varepsilon_{i+1}$ .

By induction on  $j$ , the lemma follows.  $\square$

**Lemma 3.3.** *Let  $K$  be a subset of a direct sum  $X \oplus_Z Y$  endowed with a strictly monotone norm. Under the assumptions of Lemma 3.1, for every positive integer  $k$ , there exist a sequence  $(\varepsilon_n(k))$  in  $(0, 1)$ , a subsequence  $(v_n(k)) = (x_n(k), y_n(k))$  of  $(w_n)$  and a family  $\{D_j^i(k)\}_{1 \leq j \leq k, i \geq 1}$  of relatively compact convex subsets of  $K$  such that  $\|b\| < \frac{1}{k}$  for every  $u = (a, b) \in \bigcup_{i=1}^{\infty} D_k^i(k)$ .*

*Proof.* Since  $Z = (\mathbb{R}^2, \|\cdot\|_Z)$  is a finite dimensional space and the norm  $\|\cdot\|_Z$  is strictly monotone, for every  $\varepsilon > 0$ , there exists  $\delta(\varepsilon) > 0$  such that if  $(\bar{a}, \bar{b}), (\bar{a}, \bar{c})$  belong to the unit ball  $B_Z$  and  $\|(\bar{a}, \bar{b})\| < \|(\bar{a}, \bar{c})\| + \delta(\varepsilon)$ , then  $\|\bar{b}\| < \|\bar{c}\| + \varepsilon$ . Fix  $k \geq 1$ ,  $\eta = \frac{1}{4k}$  and choose

$$\varepsilon_i = \varepsilon_i(k) < \min \left\{ \frac{\delta(\eta^i)}{3k}, \frac{\eta^i}{k} \right\}, \quad i \geq 1.$$

By Lemma 3.1, there exist a sequence  $(v_n(k)) = (x_n(k), y_n(k))$  and a family  $\{D_j^i(k)\}_{1 \leq j \leq k, i \geq 1}$  of relatively compact convex subsets of  $K$  with the properties described in this lemma.

Let  $u = (a, b) \in D_k^i(k)$ ,  $i \geq 2$ . It follows from Lemma 3.2 that there exists  $z = (x, y) \in D_k^{i-1}(k)$  such that

$$\|z - u\| + \|u - v_i(k)\| \leq \|z - v_i(k)\| + 3(k-1)\varepsilon_i < \|z - v_i(k)\| + 3k\varepsilon_i.$$

Hence

$$\|(\|x - x_i(k)\|, \|y - b\| + \|b - y_i(k)\|)\| < \|(\|x - x_i(k)\|, \|y - y_i(k)\|)\| + 3k\varepsilon_i$$

which yields

$$\|y - b\| + \|b - y_i(k)\| < \|y - y_i(k)\| + \eta^i.$$

Consequently,

$$\|b\| < \|y\| + \|y_i(k)\| + \frac{1}{2}\eta^i.$$

By induction with respect to  $i$ , there exists  $(\bar{x}, \bar{y}) \in D_k^1(k)$  such that

$$\|b\| < \|\bar{y}\| + (\varepsilon + \dots + \varepsilon_i) + \frac{1}{2}(\eta + \dots + \eta^i) < k\varepsilon_1 + 2\eta + \eta < 4\eta = \frac{1}{k}.$$

□

We are now in a position to prove the main theorem.

**Theorem 3.4.** *Let  $X$  be a Banach space with WFPP and  $Y$  has the GGLD property. Then  $X \oplus_Z Y$  with respect to a strictly monotone norm has WFPP.*

*Proof.* Assume that  $X \oplus_Z Y$  does not have WFPP. Then, there exist a weakly compact convex subset  $C$  of  $X \oplus_Z Y$  and a nonexpansive mapping  $T : C \rightarrow C$  without a fixed point. By the Kuratowski-Zorn lemma, there exists a convex and weakly compact set  $K \subset C$  which is minimal invariant under  $T$  and which is not a singleton. Let  $(w_n) = ((w'_n, w''_n))$  be an approximate fixed point sequence for  $T$  in  $K$ , i.e.,  $\lim_{n \rightarrow \infty} \|Tw_n - w_n\| = 0$ . Without loss of generality we can assume that  $\text{diam } K = 1$ ,  $(w_n)$  converges weakly to

$(0, 0) \in K$  and the double limit  $\lim_{n, m \rightarrow \infty, n \neq m} \|w_n - w_m\|$  exists. It follows from the Goebel-Karlovitz lemma that

$$\lim_{n, m \rightarrow \infty, n \neq m} \|w_n - w_m\| = \lim_{n \rightarrow \infty} \|w_n\| = 1. \quad (3)$$

Applying Lemma 2.1 gives  $\lim_{n \rightarrow \infty} \|w_n''\| = 0$ . Lemma 3.3 now shows that for every positive integer  $k$ , there exist a subsequence  $(v_n(k)) = (x_n(k), y_n(k))$  of  $(w_n)$  and a family  $\{D_j^i(k)\}_{1 \leq j \leq k, i \geq 1}$  of relatively compact convex subsets of  $K$  such that  $\|b\| < \frac{1}{k}$  for every  $u = (a, b) \in \bigcup_{i=1}^{\infty} D_k^i(k)$ .

Let  $C_0 = \{(0, 0)\}$  and  $C_j = \text{conv}(C_{j-1} \cup T(C_{j-1}))$  for  $j \geq 1$ . It is not difficult to see that  $\text{cl}(\bigcup_{j=1}^{\infty} C_j)$  is a closed convex subset of  $K$  which is invariant for  $T$  (and hence equals  $K$ ). Fix  $k \geq 1$  and notice that  $(0, 0) \in \text{cl}(\bigcup_{i=1}^{\infty} D_1^i(k))$ , because a sequence  $(v_n(k))_{n \geq 1}$  converges weakly to  $(0, 0)$ . Furthermore,

$$T(\text{cl}(\bigcup_{i=1}^{\infty} D_j^i(k))) = \text{cl}(\bigcup_{i=1}^{\infty} T(D_j^i(k))) \subset \text{cl}(\bigcup_{i=1}^{\infty} D_{j+1}^i(k))$$

and hence, by induction on  $j$ ,

$$C_j \subset \text{cl}(\bigcup_{i=1}^{\infty} D_{j+1}^i(k)) \subset \text{cl}(\bigcup_{i=1}^{\infty} D_k^i(k)), \quad j < k.$$

It follows that if  $(x, y) \in C_j$  and  $j < k$ , then  $\|y\| \leq \frac{1}{k}$ . Since  $k$  is arbitrary,  $y = 0$  for every  $(x, y) \in \text{cl}(\bigcup_{j=1}^{\infty} C_j) = K$ . Therefore,  $K$  is isometric to a subset of  $X$ . Since  $X$  has WFPP,  $T$  has a fixed point in  $K$ , which contradicts our assumption.  $\square$

#### 4. CONSEQUENCES

In this section, we list some consequences of Theorem 3.4. Notice that in the case of reflexive spaces, the properties FPP and WFPP coincide. Furthermore, if a Banach space  $Y$  has uniform normal structure ( $N(Y) > 1$ ), then  $Y$  is reflexive and has FPP. In the remainder of this section,  $X \oplus_Z Y$  denotes a direct sum of Banach spaces  $X$  and  $Y$  with respect to a strictly monotone norm.

**Corollary 4.1.** *Suppose  $X$  is a reflexive Banach space with FPP and  $Y$  has uniform normal structure. Then  $X \oplus_Z Y$  has FPP.*

In particular, the above corollary is valid if  $X$  is a uniformly nonsquare or a uniformly noncreasy Banach space.

**Corollary 4.2.** *Suppose  $X$  is a Banach space with WFPP and  $Y$  satisfies Bynum's condition  $\text{WCS}(Y) > 1$ . Then  $X \oplus_Z Y$  has WFPP.*

It is well known that all finite dimensional spaces have uniform normal structure. A very particular case of the above corollary answers a question of M. A. Khamsi (see [20, p. 999]) for strictly monotone norms.

**Corollary 4.3.** *Suppose  $X$  is a Banach space with WFPP and  $Y$  is a finite dimensional space. Then  $X \oplus_Z Y$  has WFPP.*

A Banach space  $X$  with the property that  $X \oplus_1 \mathbb{R}$  has WFPP has been studied in [22]. The following theorem was established for the  $\ell_1^2$ -norm but the proof is valid for all strictly monotone norms.

**Theorem 4.4** (see [22, Theorem 1]). *Suppose  $X$  is a Banach space such that  $X \oplus_Z \mathbb{R}$  has WFPP. Let  $Y$  be a Banach space which is uniformly convex in every direction. Then  $X \oplus_Z Y$  has WFPP.*

Corollary 4.3 and Theorem 4.4 give the following result.

**Theorem 4.5.** *Suppose  $X$  is a Banach space with WFPP and  $Y$  is uniformly convex in every direction. Then  $X \oplus_Z Y$  has WFPP.*

Recall that the James space  $J$  is an example of a Banach space with the GGLD property which is not uniformly convex in every direction and the space  $c_0$  with the norm

$$\|x\| = \sqrt{\|x\|_\infty^2 + \sum_{i=1}^{\infty} \frac{x_i^2}{2^i}}$$

is an example of a Banach space which is uniformly convex in every direction but fails the GGLD property (see [11] and references therein). This shows that Theorem 3.4 and Theorem 4.5 are independent of each other.

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## REFERENCES

- [1] D. E. Alspach, A fixed point free nonexpansive map, Proc. Amer. Math. Soc. 82 (1981), 423–424.
- [2] J. M. Ayerbe Toledano, T. Domínguez Benavides, G. López Acedo, Measures of Noncompactness in Metric Fixed Point Theory, Birkhäuser Verlag, Basel, 1997.
- [3] L. P. Belluce, W. A. Kirk, E. F. Steiner, Normal structure in Banach spaces, Pacific J. Math. 26 (1968), 433–440.
- [4] A. Betiuk-Pilarska, S. Prus, Banach lattices and the fixed point property for direct sums, in: Fixed Point Theory and its Applications, S. Dhompongsa et al. (eds.), Yokohama Publ., Yokohama, 2008.
- [5] A. Betiuk-Pilarska, S. Prus, Uniform nonsquareness of direct sums of Banach spaces, Topol. Methods Nonlinear Anal. 34 (2009), 181–186.
- [6] S. Dhompongsa, A. Kaewcharoen, A. Kaewkhao, Fixed point property of direct sums, Nonlinear Anal. 63 (2005), e2177–e2188.
- [7] S. Dhompongsa, S. Saejung, Geometry of direct sums of Banach spaces, Chamchuri J. Math. 2 (2010), 1–9.
- [8] T. Domínguez Benavides, A renorming of some nonseparable Banach spaces with the fixed point property, J. Math. Anal. Appl. 350 (2009), 525–530.
- [9] P. N. Dowling, P. K. Lin, B. Turett, Direct sums of renormings of  $\ell_1$  and the fixed point property, Nonlinear Anal. 73 (2010), 591–599.
- [10] P. N. Dowling, S. Saejung, Extremal structure of the unit ball of direct sums of Banach spaces, Nonlinear Anal. 68 (2008), 951–955.
- [11] J. García Falset, E. Lloréns Fuster, Normal structure and fixed point property, Glasgow Math. J. 38 (1996), 29–37.



- [12] J. García Falset, E. Lloréns Fuster, E. M. Mazcuñan Navarro, Uniformly nonsquare Banach spaces have the fixed point property for nonexpansive mappings, *J. Funct. Anal.* 233 (2006), 494–514.
- [13] K. Goebel, On the structure of minimal invariant sets for nonexpansive mappings, *Ann. Univ. Mariae Curie-Skłodowska Sect. A* 29 (1975), 73–77.
- [14] K. Goebel, W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge, 1990.
- [15] *Handbook of Metric Fixed Point Theory*, W. A. Kirk, B. Sims (eds.), Kluwer Academic Publishers, Dordrecht, 2001.
- [16] A. Jiménez-Melado, Stability of weak normal structure in James quasi reflexive space, *Bull. Austral. Math. Soc.* 46 (1992), 367–372.
- [17] L. A. Karlovitz, Existence of fixed points of nonexpansive mappings in a space without normal structure, *Pacific J. Math.* 66 (1976), 153–159.
- [18] M. Kato, K.-S. Saito, T. Tamura, *Uniform non-squareness of  $\psi$ -direct sums of Banach spaces  $X \oplus_{\psi} Y$* , *Math. Inequal. Appl.* 7 (2004), 429–437.
- [19] M. Kato, T. Tamura, Weak nearly uniform smoothness and worth property of  $\psi$ -direct sums of Banach spaces, *Comment. Math. Prace Mat.* 46 (2006), 113–129.
- [20] M. A. Khamsi, On normal structure, fixed point property and contractions of type  $\gamma$ , *Proc. Amer. Math. Soc.* 106 (1989), 995–1001.
- [21] W. A. Kirk, A fixed point theorem for mappings which do not increase distances, *Amer. Math. Monthly* 72 (1965), 1004–1006.
- [22] T. Kuczumow, S. Reich, M. Schmidt, A fixed point property of  $l_1$ -product spaces, *Proc. Amer. Math. Soc.* 119 (1993), 457–463.
- [23] T. Landes, Permanence properties of normal structure, *Pacific J. Math.* 110 (1984), 125–143.
- [24] T. Landes, Normal structure and the sum-property, *Pacific J. Math.* 123 (1986), 127–147.
- [25] P. K. Lin, There is an equivalent norm on  $\ell_1$  that has the fixed point property, *Nonlinear Anal.* 68 (2008), 2303–2308.
- [26] G. Marino, P. Pietramala, H. K. Xu, Geometrical conditions in product spaces, *Nonlinear Anal.* 46 (2001), 1063–1071.
- [27] B. Maurey, Points fixes des contractions de certains faiblement compacts de  $L^1$ , *Semin. Anal. Fonct.* 1980-1981, Ecole Polytechnique, Palaiseau, 1981.
- [28] S. Prus, A. Wiśnicki, On the fixed point property in direct sums of Banach spaces with strictly monotone norms, *Studia Math.* 186 (2008), 87–99.
- [29] B. Sims, M. A. Smyth, On some Banach space properties sufficient for weak normal structure and their permanence properties, *Trans. Amer. Math. Soc.* 351 (1999), 497–513.
- [30] A. Wiśnicki, On the super fixed point property in product spaces, *J. Funct. Anal.* 236 (2006), 447–456, corrigendum, *ibid.*, 254 (2008), 2313–2315.

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